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Implementation of a three-qubit refined Deutsch–Jozsa algorithm using SFG quantum logic gates

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Abstract

In this paper we present a quantum logic circuit which can be used for the experimental demonstration of a three-qubit solid state quantum computer based on a recent proposal of optically driven quantum logic gates. In these gates, the entanglement of randomly placed electron spin qubits is manipulated by optical excitation of control electrons. The circuit we describe solves the Deutsch problem with an improved algorithm called the refined Deutsch–Jozsa algorithm. We show that it is possible to select optical pulses that solve the Deutsch problem correctly, and do so without losing quantum information to the control electrons, even though the gate parameters vary substantially from one gate to another.

1. Introduction

In recent years, many different physical systems have been proposed to implement a quantum computer. However, the task of building a large-scale quantum computer is still unresolved. The problems are of two sorts: those relating to the hardware available for quantum operations, and those associated with the algorithms and their implementation as sequences of controlled manipulations. As regards hardware, given the substantial expertise acquired from the research area of microelectronics and because quantum information processors will need to be integrated within classical digital microelectronics, localized spins in solids, or, specifically, in semiconductors, appear as promising candidates [1]. Contrary to previous schemes [2], which used metallic gate electrodes and required very low operational temperatures in order to avoid the ionization of defects close to the gates, in this work we analyse the potential of a scheme which avoids electrodes on the chip and may be able to work at liquid nitrogen or even near room temperature [1, 3]. In this scheme, qubits are carried by electron spins of randomly placed deep donors, for example two particles A and B, with spacing between them large enough to have small ground-state interactions. However, the qubits can be made to interact by the controlled optical excitation of a 'control' electron from a nearby control particle C into a molecular state of A and B. In terms of their wavefunctions, the system is

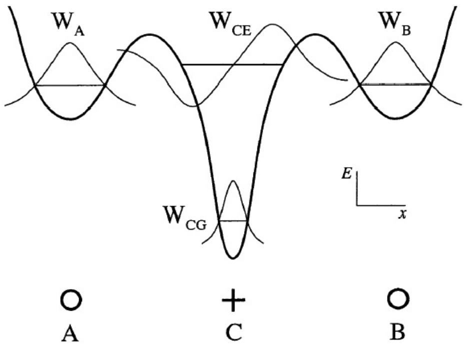

Figure 1. Particles A and B are separated by a distance large enough for their wavefunctions W_{A} and W_{B} not to interact. When the control electron on C is excited by an optical pulse, its wavefunction W_{CE} overlaps with W_{A} and W_{B}, allowing them to interact [1].

schematically shown in figure 1 [1]. The interaction between the qubits is, therefore, switched on by the excitation of the control electron through an optical pulse and switched off again by the transmission of a second optical pulse which de-excites the control electron back to its ground-state. For simplicity, we will refer to these gates as to SFG gates (from the initials of the authors of this quantum logic gate scheme [1]) in the remainder of this paper. We shall discuss the sequence of optical pulses that would be needed to implement a specific algorithm on a quantum computer based on SFG gates. In analysing the potential of these two-qubit quantum logic gates, we examine a quantum logic circuit suitable to demonstrate a three-qubit quantum computer prototype. In particular, we shall demonstrate that sensible pulse sequences could be devised to implement the refined Deutsch–Jozsa algorithm presented in [4]. This is an improved version of the algorithm which solves the Deutsch problem on n bits using a quantum register of n qubits as opposed to the original version presented by Deutsch and Jozsa [5, 6] which required $n + 1$.

The algorithm is based on a mathematical problem which, because of its flexibility (it can be implemented for any number of qubits), has been used for the experimental demonstration of many quantum computer prototypes [7–9]. We shall analyse the physical parameters which lead to good approximations of controlled phase gates [10], a convenient gate for the implementation of the Deutsch–Jozsa algorithm. As a further test, we use numerical simulation to compare the solution of the algorithm obtained from an ideal quantum logic circuit with the solution based on SFG gates. Thus our investigation provides guidelines for the development and design of a prototype solid state quantum computer based on SFG gates.

2. SFG gates: the three-spin system

We consider three particles A, B and C in a magnetic field **B**. Particles A and B are deep donors, whose electron spins carry the qubits. Particle C, the control atom, provides the control electron which, when excited by an optical pulse, mediates the interaction of A and B. As shown in [3], this system can be analysed assuming the dominant interaction to be exchange between the control and qubit electrons, represented by effective Heisenberg interactions of strength J_{A} and J_{B}, and can be described by the Hamiltonian

$$\begin{aligned} H = {} & |g\rangle \left\{ B_{\mathrm{A}}\sigma_{\mathrm{A}z} + B_{\mathrm{B}}\sigma_{\mathrm{B}z} + B_{\mathrm{C}}^{0}\sigma_{\mathrm{C}z} \right\} \langle g| \\ & + |e\rangle \left\{ J_{\mathrm{A}}\sigma_{\mathrm{A}}\cdot\sigma_{\mathrm{C}} + J_{\mathrm{B}}\sigma_{\mathrm{B}}\cdot\sigma_{\mathrm{C}} + B_{\mathrm{A}}\sigma_{\mathrm{A}z} + B_{\mathrm{B}}\sigma_{\mathrm{B}z} + B_{\mathrm{C}}\sigma_{\mathrm{C}z} + \varepsilon \right\} \langle e| \\ & + |e\rangle\, V(t) \cos(\omega t + \phi) \langle g| + |g\rangle\, V(t) \cos(\omega t + \phi) \langle e| \,. \end{aligned} \quad (1)$$

Here $|g\rangle$ and $|e\rangle$, respectively, denote the ground and excited state of the control atom. There is a magnetic field $\mathbf{B}$, so there are Zeeman energies involving μ_A and μ_B as the magnetic moments of the qubit spins, while μ_C^0 is the ground state magnetic moment of the control particle and μ_C is its excited state magnetic moment. In expression (1), these quantities have been used to define

$$B_K = -|\mathbf{B}|\,\mu_K \quad (\mathrm{K} = \mathrm{A}, \mathrm{B}, \mathrm{C}) \qquad B_C^0 = -|\mathbf{B}|\,\mu_C^0. \tag{2}$$

The σ_i are the Pauli spin matrices. The ground and excited states are separated by the excitation energy ε, and coupled by the oscillating part of the Hamiltonian, corresponding to the optical excitation, $V(t)\cos(\omega t + \phi)$. The interaction is initiated by the excitation of the control electron through an optical pulse and terminated, but not destroyed, by the transmission of a second optical pulse after a time T. Note that the magnetic field $\mathbf{B}$ is the same for both qubits and control. Typical spacings of the dopants would be a few tens of nanometres in silicon, far less than distances on which it is easy to apply unique fields for each. There will be fields due to other magnetic species present. Following [3], all our analysis here will be made for the symmetric case $J_A = J_B = J$ and $B_A = B_B = B$. It is possible to generalize the results to less symmetric cases.

An important issue for the design of an SFG based quantum computer concerns *selectivity*, i.e. each gate must be individually addressable within a quantum register of SFG gates. Since gates are addressed through optical pulses, the frequencies ω of the excitation and de-excitation pulses for each gate must be different. As described in [1], this is possible by creating disordered arrangements of the deep donors, since disorder will make the excitation energies vary from one pair A, B to another because of the different spacings and orientations relative to the host crystallographic axes.

2.1. Gates that leave the control electron unentangled

Of all quantum logic gates, described by the unitary transformation

$$U = \mathrm{e}^{-\mathrm{i}HT} \tag{3}$$

we are especially interested in those in which the control electron remains completely unentangled from the two qubits at the end of the time interval T.

As demonstrated in [3], there are many such solutions, and these solutions can be described by means of two integers M and N. Also, since we are analysing transformations which leave the control electron unentangled from the qubits, we can describe the quantum gates by considering only the computational space spanned by the qubits, neglecting the control electron. Under these assumptions, the two-qubit quantum logic gates implementable by SFG gates are

$$U_+(M,N) = \mathrm{e}^{\mathrm{i}(J-B)T}\begin{bmatrix} \mathrm{e}^{-\mathrm{i}[(3-f)J+2B]T} & 0 & 0 & 0 \\ 0 & \frac{[(-1)^M + \mathrm{e}^{-\mathrm{i}(1-f)JT}]}{2} & \frac{[(-1)^M - \mathrm{e}^{-\mathrm{i}(1-f)JT}]}{2} & 0 \\ 0 & \frac{[(-1)^M - \mathrm{e}^{-\mathrm{i}(1-f)JT}]}{2} & \frac{[(-1)^M + \mathrm{e}^{-\mathrm{i}(1-f)JT}]}{2} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{2\mathrm{i}BT}(-1)^N \end{bmatrix}$$
$$U_-(M,N) = \mathrm{e}^{\mathrm{i}(J+B)T}\begin{bmatrix} \mathrm{e}^{-2\mathrm{i}BT}(-1)^M & 0 & 0 & 0 \\ 0 & \frac{[(-1)^N + \mathrm{e}^{-\mathrm{i}(1+f)JT}]}{2} & \frac{[(-1)^N - \mathrm{e}^{-\mathrm{i}(1+f)JT}]}{2} & 0 \\ 0 & \frac{[(-1)^N - \mathrm{e}^{-\mathrm{i}(1+f)JT}]}{2} & \frac{[(-1)^N + \mathrm{e}^{-\mathrm{i}(1+f)JT}]}{2} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\mathrm{i}[(3+f)J-2B]T} \end{bmatrix} \tag{4}$$

where U_+ and U_- are transformations which are active depending on whether the control electron starts in the spin-up or spin-down state and where we can link the physical parameters

B, J, T to parameters M, N and f by these equations:

$$\begin{aligned}
\frac{B}{J} &= f \\
f &= -\frac{M^2+N^2}{M^2-N^2} \pm \sqrt{\left(\frac{M^2+N^2}{M^2-N^2}\right)^2 - 9} \\
JT &= \frac{M\pi}{\sqrt{(f-1)^2+8}} = \frac{N\pi}{\sqrt{(1+f)^2+8}} \\
BT &= \frac{M\pi}{\sqrt{\left(1-\frac{1}{f}\right)^2+\frac{8}{f^2}}} = \frac{N\pi}{\sqrt{\left(1+\frac{1}{f}\right)^2+\frac{8}{f^2}}}.
\end{aligned} \tag{5}$$

The exchange interaction J depends on the distribution of the particles and decreases with increasing distance between the qubits and the control atoms. The distance between particles has to be large enough to avoid ground-state interactions between the qubits. As shown in [3], this requirement leads to distances between qubits and control atoms of about 13–17 nm and corresponding values of the exchange interaction J around 1–20 GHz. f needs to be real and T positive, which implies M and N to be positive and to be bound by the following constraints:

$$\frac{M}{\sqrt{2}} \leqslant N \leqslant \sqrt{2}M \tag{6}$$

for solutions of f with the positive root, and

$$\left\{\frac{M}{\sqrt{2}} \leqslant N \leqslant \sqrt{2}M\right\} \cap \{M \neq N\} \tag{7}$$

for solutions of f with the negative root, as, in this case, f diverges for $M = N$. As described in [3], typically, $JT \sim 1000$, which leads, as will be shown later, to values of M and N of a few hundred to a few thousand.

Another parameter which is fundamental for the description of a realistic SFG gate is the pulse length τ of the optical exciting and de-exciting pulses. The pulse length τ needs to be short enough to have a bandwidth which covers all spin components of the excited state; with realistic values, the pulse length is expected to be of the order of a few ps [3]. Hence, while the strength of the exchange interaction J depends on the relative geometrical position of the particles A, B and C, and represents, therefore, a fixed parameter of the experimental set-up, the control parameters of the gate are the magnetic field **B** and the pulse character, given by the interval T between pulses and the pulse length τ. As mentioned above, for a chip comprising many SFG gates, the magnetic field **B** is a parameter which will have to be the same for each gate, apart from the effects of other magnetic species present.

3. The refined Deutsch–Jozsa algorithm

The Deutsch problem addresses the following scenario.

A function $f(x)$ which takes as input a variable x expressed on n bits is assumed. If the output, a single bit, is always 0 or always 1, independent of the value of x, then the function is called *constant*. If the output is 0 for exactly half of all possible input values and 1 for the remaining ones, then the function is called *balanced*. Suppose further that an oracle makes a random selection between constant or balanced functions $f(x)$. How many queries of the function are necessary to determine the type of function (constant or balanced) the oracle has chosen?

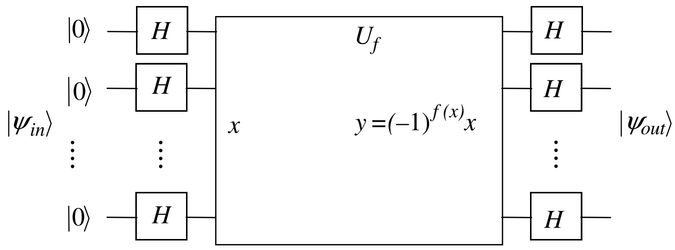

Figure 2. 'Wire-diagram' representation of the quantum logic circuit for the refined Deutsch–Jozsa algorithm. The diagram is to be read from left to right and each wire represents a qubit.

A quantum circuit which solves the Deutsch problem is shown in figure 2 [4]. This solution is known as the refined Deutsch–Jozsa algorithm [4, 9] since it requires a quantum register of n qubits whereas the original version presented by Deutsch and Jozsa [5, 6] required $n + 1$. The representation of the quantum circuit given in figure 2 is known as the 'wire-diagram' of the circuit [10]. This representation is to be read from left to right, hence, the left side of the diagram represents the input of the quantum circuit, which, in this case, starts with each qubit being in the state $|0\rangle$. One can follow the quantum logic operations applied to a qubit by analysing the wire which starts from its initial state. Quantum operations which only change the state of a single qubit are symbolized by a box centred on a single wire with a code characterizing the type of transformation. Similarly, a multi-qubit operation is described by a box connecting the two or more qubits involved in the quantum transformation.

From figure 2, for example, one can see that the first operation applied to each qubit is a *Hadamard* gate, i.e. a single-qubit operation described mathematically by

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{8}$$

Then, the quantum transformation $U_f = (-1)^{f(x)}$ is applied to the whole quantum register. Finally, the output state $|\psi_{\text{out}}\rangle$ is obtained applying a final *Hadamard* gate H to each qubit. A measurement on the output register of the circuit shown in figure 2 gives all 0s if the oracle had chosen a constant function, and gives at least a 1 for the case of a balanced function. Hence, applying U_f once and with one single measurement only, it is possible to determine the type of function chosen by the oracle, whereas with a classical computer one would need $2^{n-1} + 1$ queries of the function $f(x)$ in the worst case.

The core of the of the Deutsch–Jozsa algorithm is in the implementation of the unitary transformation $U_f = (-1)^{f(x)}$. Because of its simplicity and its flexibility (it can be implemented for any number of qubits) the Deutsch–Jozsa algorithm has been used for many experimental demonstrations of quantum computers [7–9].

As described in [4] it is a good candidate to test a quantum computer prototype since it exploits the three important features of quantum computation: superposition, interference and entanglement, although entanglement is created between the qubits only for $n \geqslant 3$ and for some specific balanced functions.

In [9], quantum logic circuits for a three-qubit NMR quantum computer are reported. In particular, [9] gives the gate sequences for implementing the unitary transformations corresponding to all 35 nontrivial balanced functions. There are 70 balanced functions in total; however, since the quantum transformations corresponding to $f(x)$ and $\bar{f}(x) = 1 - f(x)$ only differ by a global phase-shift, they are described by the same circuit [10], hence the total of 35 nontrivial quantum circuits. The sequences are subdivided in groups depending on how many two-qubit gates they require for their implementation. A maximum of three two-qubit gates is necessary to implement any of the 35 functions. For example, let us consider the following balanced function, which requires three two-qubit gates:

$$\begin{array}{ll} f(000)=0 & f(001)=0 \\ f(010)=0 & f(011)=1 \\ f(100)=0 & f(101)=1 \\ f(110)=1 & f(111)=1. \end{array} \tag{9}$$

Using the notation described in [9], we label this function using the following method: first, the outputs of the function are ordered with respect to the ascending binary values of their inputs. For the function described by expression (9), we obtain the string 00010111. The function is labelled using the hexadecimal value corresponding to the binary string. As the corresponding hexadecimal value of our string is 17, we label this specific balanced function f_{17}.

As shown in [9], the following gate sequence implements f_{17}:

$$U_{f_{17}} = R_{z1}(\pi)\, J_{21}\left(\frac{\pi}{2}\right) J_{10}\left(\frac{\pi}{2}\right) J_{20}\left(-\frac{\pi}{2}\right) \tag{10}$$

where $R_{zi}(\vartheta)$ represents a single-qubit operation on qubit i which rotates a spin of ϑ degrees around the z-axis in the Bloch-sphere representation [10]. $J_{ij}(\vartheta)$ describes the two-qubit interaction between qubits i and j and is given mathematically by

$$J_{ij}(\vartheta) = \mathrm{e}^{-\mathrm{i}\frac{\vartheta}{2}\sigma_{zi}\cdot\sigma_{zj}} \tag{11}$$

with σ_{zi} being the Pauli z matrix for the ith qubit expressed in the computational space spanned by the qubits.

We now insert the gate sequence of expression (10) in the circuit described in figure 2. This gives for the state of the quantum register at the end of the whole gate sequence, but prior to the measurement:

$$|\psi_{\text{out}-\text{ideal}}\rangle = -0.3536 \times \begin{bmatrix} 0 \\ 1+i \\ 1+i \\ 0 \\ 1+i \\ 0 \\ 0 \\ -(1+i) \end{bmatrix}. \tag{12}$$

The notation used is such that the first element of the column vector indicates the probability amplitude for the state $|0\rangle|0\rangle|0\rangle$, the second element corresponds to the state $|0\rangle|0\rangle|1\rangle$ and continues in ascending binary order up to the last probability amplitude which corresponds to the state $|1\rangle|1\rangle|1\rangle$. The subscript 'ideal' indicates that equation (12) assumes ideal gates. Real gates are subject to constraints, and we shall use these results in comparing them with the ideal gates. It is important to note that the output state given in expression (12) correctly solves the Deutsch problem since, as summarized above, the implemented function is balanced and only states with at least one qubit in the $|1\rangle$-state have a non-zero probability of being measured.

4. Solving the algorithm using SFG gates

It is helpful to relate the quantum logic gates of the last section to standard gates, and specifically to the controlled phase gate and to operations on single qubits. Both $J_{ij}(\frac{\pi}{2})$ and

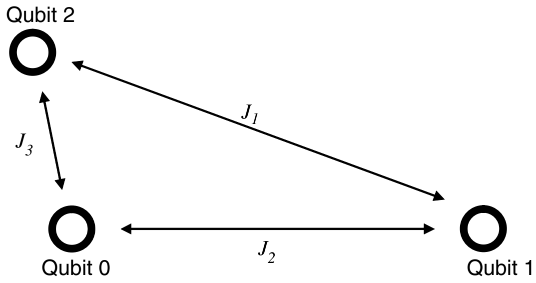

Figure 3. Three-qubit SFG gate system.

$J_{ij}(-\frac{\pi}{2})$ are quantum logic gates which are locally equivalent [11] to the controlled phase gate $U_{C-\text{PHASE}}$:

$$U_{C-\text{PHASE}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{13}$$

This means that it is possible to express $J_{ij}(\frac{\pi}{2})$ or $J_{ij}(-\frac{\pi}{2})$ through $U_{C-\text{PHASE}}$ and some single-qubit operations.

This has an important consequence. Consider a three-qubit chip on which it is possible to implement only SFG gates which are locally equivalent to the controlled-phase gate $U_{C-\text{PHASE}}$. We could then adapt the circuit discussed above to the case of an SFG gate-based quantum computer, and this could be done without increasing the total number of two-qubit quantum logic gates.

We assume the availability of a chip with a convenient distribution of qubits and control atoms for a three-qubit quantum register. The situation is schematically shown in figure 3, where control atoms have been omitted for simplicity.

As summarized in section 2.1, one of the parameters which characterizes SFG gates is the strength of the exchange interaction J of the qubit–control atom–qubit symmetric three-particle system. The assumption of having each pair of qubits symmetrically distributed about its control atom simplifies the analysis, but is not essential. In figure 3, we distinguish the three different SFG gates by labelling them with the strength of the exchange interaction J_i ($i = 1, 2, 3$) of the three different qubit–control atom–qubit systems. Since the quantum register has random qubits, the values of J_i will be different, and the excitation energies for the control electrons for each gate will be different [1].

Let us take the gate characterized by J_3 as a reference. As described in [3], a controlled phase gate can be implemented by adopting parameters $M = 1584$ and $N = 2177$. For these parameter values one obtains $f = 4.5$. Taking numbers consistent with the ones used in [3], let us consider $J_3 = 16$ GHz. Since

$$f = \frac{B}{J} \tag{14}$$

and since we have fixed J_3, we can compute the magnetic field term B necessary to obtain the desired gate. In this case, the magnetic field term is $B = 72$ GHz. This magnetic field term is now experienced by the whole system; in particular, it will be the same for the other SFG gates with their different values of J_i.

We can now determine which values of J_i would approximate a controlled-phase gate to a desired precision. To do this, we evaluated numerically the strength interaction J and the

Table 1. Parameters for phase gates for $B = 72$ GHz, $\Delta c_i = \pm 0.01$.

M	N	f	J
1458	1589	22.91	3.1428
1436	1566	22.74	3.1663
815	904	18.89	3.8116
797	885	18.684	3.8535
1936	2165	17.448	4.1265
1521	1716	16.101	4.4717
2133	2457	13.574	5.3042
1873	2170	12.994	5.5412
637	746	12.018	5.991
1214	1436	11.218	6.418
1509	1788	11.09	6.4922
1171	1406	10.172	7.078
1897	2297	9.6475	7.4631
1597	1998	7.9437	9.0638
1931	2429	7.7005	9.3501
1295	1657	6.9897	10.301
473	624	5.8691	12.268
1419	1872	5.8691	12.268
1525	2034	5.4983	13.095
1595	2137	5.348	13.463
1146	1540	5.2495	13.716
1732	2380	4.5059	15.979
1584	2177	4.5	16
437	601	4.477	16.082
1615	2274	3.4977	20.585
679	934	2.0133	35.762
1553	2136	2.0116	35.792
437	601	2.0103	35.816
1584	2177	2	36

entangling characteristics of all SFG gates for M and N values lying between 1 and 2500 and $B = 72$ GHz. In other words, we analysed which SFG gates, for the magnetic field term evaluated above, have c_1, c_2 and c_3 coefficients described in [11] with values in the vicinity of those expected for a controlled phase gate within a given tolerance $\pm\Delta c_i$. Table 1 shows which other combinations of parameters approximate the phase gate given by $M = 1584$ and $N = 2177$.

From table 1 it can be seen that there are many combinations which allow us to implement controlled-phase gates. Selecting out of this set values of J_i similar to the ones shown in [3], a possible set of SFG gates implementing controlled-phase gates for a three-qubit system could be the following.

SFG gate 1: between qubit 0 and 2. $J_1 = 3.8116$ GHz, $M = 815$ and $N = 904$, $JT = 141.3636$, $T_1 = 37.088$ ns.

SFG gate 2: between qubit 0 and 1. $J_2 = 13.463$ GHz, $M = 1595$ and $N = 2137$, $JT = 966.0433$, $T_2 = 71.752$ ns.

SFG gate 3: between qubit 1 and 2. $J_3 = 16$ GHz, $M = 1584$ and $N = 2177$, $JT = 1105.84$, $T_3 = 69.115$ ns.

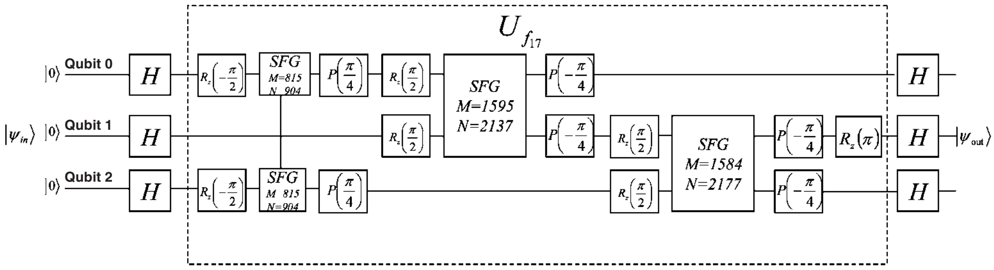

Figure 4. Total circuit for the refined Deutsch–Jozsa algorithm with a three-qubit SFG quantum computer for the case of function f_{17}.

As shown in [10], $J_{ij}(\pm\frac{\pi}{2})$ (note that this transformation should not be confused with the interaction strength J and J_i) can be expressed through $U_{C-\text{PHASE}}$ with the following circuits:

$$
\begin{aligned}
J_{ij}\left(-\frac{\pi}{2}\right) &= P\left(\frac{\pi}{4}\right) U_{C-\text{PHASE}ij} R_{zi}\left(-\frac{\pi}{2}\right) R_{zj}\left(-\frac{\pi}{2}\right) \\
J_{ij}\left(\frac{\pi}{2}\right) &= P\left(-\frac{\pi}{4}\right) U_{C-\text{PHASE}ij} R_{zi}\left(\frac{\pi}{2}\right) R_{zj}\left(\frac{\pi}{2}\right)
\end{aligned}
\tag{15}
$$

where $P(\vartheta) = e^{-i\vartheta}$ is simply a constant phase shift and the subscripts i and j indicate to which qubits the transformations are applied. Using expressions (10), (15) and figure 2, the complete circuit which solves the refined Deutsch–Jozsa algorithm using SFG gates for the case of function f_{17} is shown in figure 4.

Again, the circuit shown in figure 4 is given in the 'wire-diagram' representation. Compared to figure 2, the quantum operation U_f has been decomposed into a sequence of quantum logic gates involving only single-qubit operations and two-qubit SFG gates which implement the transformation $U_{f_{17}}$. Each SFG gate connects vertically two qubits and is characterized by the couple of integers M and N which define its parameters, as described through the equations given in (5). Single-qubit operations are symbolized by a box centred on a single wire and are either *Hadamard* gates H, rotations around the z-axis $R_{zi}(\vartheta)$ or constant phase-shifts $P(\vartheta)$.

Simulating the performance of the circuit, which means to apply to the quantum register starting in the state $|\psi_{\text{in}}\rangle$ all the transformations shown in figure 4, gives

$$
|\psi_{\text{out}-\text{SFG}}\rangle = -0.3536 \times \begin{bmatrix} -0.01 + i0.01 \\ 1 + i \\ 1 + i0.99 \\ 0.01 - i0.01 \\ 0.99 + i1.01 \\ -0.02 + i0.02 \\ 0 \\ -(1 + i) \end{bmatrix}. \tag{16}
$$

The two output states $|\psi_{\text{out}-\text{ideal}}\rangle$ and $|\psi_{\text{out}-\text{SFG}}\rangle$ can be compared by computing the fidelity, defined as [12]

$$
\text{fidelity} = |\langle\psi_{\text{ideal}}|\psi_{\text{err}}\rangle|^2 \tag{17}
$$

which is unity for parallel states and zero for orthogonal ones. Note that, while the output states given in expressions (12) and (16) have been truncated for simplicity, the fidelity has been evaluated with the non-truncated version of the output states. The final result is

$$
|\langle\psi_{\text{out}-\text{ideal}}|\psi_{\text{out}-\text{SFG}}\rangle|^2 = 0.999\,834. \tag{18}
$$

This shows that the output state obtained through the SFG controlled phase gates is a rather precise approximation of the ones obtained using the ideal quantum logic gates. The SFG quantum circuit requires six optical pulses for the three two-qubit interactions; it additionally needs two single-qubit operations and two constant phase shifts for each two-qubit interaction in order to have a controlled phase gate, plus a final single qubit operation for implementing U_f and the six single qubit operations for the *Hadamard* gates at the beginning and end of the algorithm. This gives a total of six optical pulses, 13 single qubit operations and six constant phase shifts.

Finally, we can obtain an estimate of the total computational time by summing the time intervals in which the two-qubit gates are active. Since the circuit requires one two-qubit gate between each pair of qubits, the total time is

$$T_{\mathrm{COMP}} = T_1 + T_3 + T_3 = 37.088 \text{ ns} + 71.752 \text{ ns} + 69.115 \text{ ns} = 177.36 \text{ ns}. \tag{19}$$

Just to get an idea about the impact of decoherence, let us compare this result with some recently published data about decoherence in silicon systems with qubits carried by the electron spins of phosphorus atoms [13]. This is not the only possible system; another candidate which might have promising decoherence times is bismuth-doped silicon [3]. As shown in [13], in the case of phosphorus-doped silicon, one can expect decoherence times of the order of some milliseconds. Since the estimated computational time T_{COMP} is several orders of magnitude smaller than this, we can expect our circuit to work in a regime in which the impact of decoherence is tolerable, as described by the third of di Vincenzo's requirements for quantum computation [14]. This result, combined with the high fidelity with which our circuit approximates the ideal solution of the Deutsch–Jozsa algorithm, confirms how quantum information processing based on solid-state systems seems increasingly feasible.

Clearly, the impact of decoherence requires further studies and understanding. Details of the main decoherence mechanisms which could affect the qubit–control atom–qubit system can be found in [3]. In terms of the impact of decoherence on the final result of the computation, it will be necessary to investigate what kind of influence decoherence can have on the gate dynamics and, therefore, on the choice of the gate parameters T, M and N, in the case of more complex and time consuming algorithms. In SFG gate-based systems decoherence acts both on the qubits and on the control electrons. When qubits are affected by decoherence, then the state of the quantum register is altered, which corresponds to a perturbation of the information which needs to be processed. Further, since control electrons are responsible for the implementation of the two-qubit quantum logic gates, if their state is perturbed by decoherence, then the quantum transformations one wants to implement will be perturbed as well. In both cases, errors are introduced in the computation. A detailed analysis of this phenomenon will, therefore, be the object of future studies.

5. Conclusions

We analysed the implementation of a three-qubit refined Deutsch–Jozsa algorithm for a quantum computer based on SFG gates. We showed that an accurate solution can be obtained by using SFG gates to simulate controlled-phase gates between all pairs of qubits. In a real prototype device this ideal situation is unlikely to happen as qubits and control atoms will be distributed randomly. However, the parameters presented in table 1 show that there is some degree of freedom, and it should be possible to come close to such an ideal scenario. In the worst case, this would mean that we could expect an increase of a factor of three in the total number of two-qubit quantum gates [15].

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